

On A_k -singularity on a plane curve of fixed degree.

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There is a general problem to describe singularities which can be met on algebraic hypersurfaces, in particular on plane curves, of fixed degree (see, e.g., [2]). Here we shall consider A_k -singularities which can be met on a plane curve of degree d . Let $k(d)$ be the maximal possible integer k such that there exists a plane curve of degree d with an A_k -singularity. Statement 1 gives an upper bound for $k(d)$. According to it $\overline{\lim}_{d \rightarrow \infty} k(d)/d^2 \leq 3/4$. We construct a plane curve of degree $28s + 9$ ($s \in \mathbb{Z}_{\geq 0}$) which has an A_k -singularity with $k = 420s^2 + 269s + 42$. Therefore one has $\underline{\lim}_{d \rightarrow \infty} k(d)/d^2 \geq 15/28$ (pay attention that $15/28 > 1/2$). The example is constructed basically in the same way as a curve of degree 22 with an A_{257} singularity in [3] (the aim of that example was somewhat different).

Statement 1 $k(d) \leq (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor \cdot \left(\left\lfloor \frac{d}{2} \right\rfloor - 1 \right).$

Remark. We believe that this statement is known, however we have not found a reference for it.

Proof. Without any loss of generality one can suppose that the curve is reduced. Therefore one can choose the infinite line in the complex projective plane \mathbb{CP}^2 so that it intersects the curve at d different points. Let the affine part of the curve d be given by the equation $\{f(x, y) = 0\}$ (where f is a polynomial of degree d). The surface $\{f(x, y) + z^2 = \varepsilon\} \subset \mathbb{C}^3$ is nonsingular and the negative inertia index of the intersection form on its second homology group $H_2(V; \mathbb{R})$ just coincides with the right hand side of the inequality in the statement (this follows, e.g., from the result of J.Steenbrink on the intersection form for quasihomogeneous singularities ([4]), applied to the (isolated) singularity $f_d(x, y) + z^2$, where $f_d(x, y)$ is the homogeneous part

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of degree d of the polynomial f . Now the statement follows from the facts that the intersection form of the A_k -singularity of 3 variables is negative defined and, if the considered curve has an A_k -singularity, then the vanishing homology group of this singularity is embedded into the homology group $H_2(V; \mathbb{R})$ (see, e.g., [1]). \square

Statement 2 *There exists a plane curve of degree $28s + 9$ which has an A_k -singularity with $k = 420s^2 + 269s + 42$ ($s \in \mathbb{Z}_{\geq 0}$).*

Proof. Let $\ell = 3s + 1$, $m = 7s + 2$, and let an (affine plane) curve C be given by the equation

$$F(x, y) = y^2 - 2yA(x, y) + x^{8\ell} + 4x^{7\ell}y^m = 0, \quad (1)$$

where $A(x, y) = [x^{4\ell} + 2x^{3\ell}y^m - 2x^{2\ell}y^{2m} + 4x^\ell y^{3m} - 10y^{4m}]$. The degree d of the curve C is equal to $4m + 1 = 7\ell + m = 28s + 9$. Let $z = y - A(x, y)$ (x and z are local coordinates on the plane near the origin). Then

$$F(x, y) = z^2 + 56x^{3\ell}y^{5m} - 56x^{2\ell}y^{6m} + 80x^\ell y^{7m} - 100y^{8m}. \quad (2)$$

Since $z(x, y) = y - x^{4\ell} +$ terms of higher degree, one has $y(x, z) = z + x^{4\ell} +$ terms of higher degree. Substituting this expression into (2) one gets

$$F(x, z) = z^2 + 56x^{3\ell+20\ell m} + \sum a_{ij}x^i z^j = z^2 + 56x^{k+1} + \sum a_{ij}x^i z^j,$$

where $k = 420s^2 + 269s + 42$ and the sum contains only members with powers (i, j) which lie over the segment $(0, 2)(k+1, 0)$, i.e., those for which $i/(k+1) + j/2 > 1$. This proves the statement. \square

Remark. There exists a similar problem to determine the maximal number $k' = k'(d)$ such that the singularity $A_{k'}$ is an adjacent to an isolated homogeneous singularity of degree d (or (what is equivalent) to a singularity from the same stratum $\mu = \text{const}$). One has $k'(d) \geq k(d)$. All the reasonings above are valid for this problem as well and thus the inequalities of the statements 1 and 2 hold also for the number $k'(d)$.

References

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